Exercise 75

When a foreign object lodged in the trachea (windpipe) forces a person to cough, the diaphragm thrusts upward causing an increase in pressure in the lungs. This is accompanied by a contraction of the trachea, making a narrower channel for the expelled air to flow through. For a given amount of air to escape in a fixed time, it must move faster through the narrower channel than the wider one. The greater the velocity of the airstream, the greater the force on the foreign object. X rays show that the radius of the circular tracheal tube contracts to about two-thirds of its normal radius during a cough. According to a mathematical model of coughing, the velocity v of the airstream is related to the radius r of the trachea by the equation

$$v(r) = k(r_0 - r)r^2 \qquad \frac{1}{2}r_0 \le r \le r_0$$

where k is a constant and r_0 is the normal radius of the trachea. The restriction on r is due to the fact that the tracheal wall stiffens under pressure and a contraction greater than $\frac{1}{2}r_0$ is prevented (otherwise the person would suffocate).

(a) Determine the value of r in the interval $\left[\frac{1}{2}r_0, r_0\right]$ at which v has an absolute maximum. How does this compare with experimental evidence?

(b) What is the absolute maximum value of v on the interval?

(c) Sketch the graph of v on the interval $[0, r_0]$.

Solution

To find the extreme values of $v(r)$ on the interval $r_0/2 \le r \le r_0$, take the derivative.

$$\begin{aligned} v'(r) &= \frac{d}{dr}[k(r_0 - r)r^2] \\ &= \frac{d}{dr}(kr_0r^2 - kr^3) \\ &= \frac{d}{dr}(kr_0r^2) - \frac{d}{dr}(kr^3) \\ &= 2kr_0r - 3kr^2 \end{aligned}$$

Then set $v'(r) = 0$ and solve for r.

$$2kr_0r - 3kr^2 = 0$$

$$r(2kr_0 - 3kr) = 0$$

$$r = 0 \quad \text{or} \quad 2kr_0 - 3kr = 0$$

$$r = 0 \quad \text{or} \quad r = \frac{2}{3}r_0$$

$r = (2/3)r_0$ is within the interval $r_0/2 \le r \le r_0$, so evaluate the function here.

$$v\left(\frac{2}{3}r_0\right) = k\left[r_0 - \left(\frac{2}{3}r_0\right)\right]\left(\frac{2}{3}r_0\right)^2 = \frac{4kr_0^3}{27} \approx 0.148kr_0^3 \qquad \text{(absolute maximum)}$$

Evaluate the function at the endpoints of the interval.

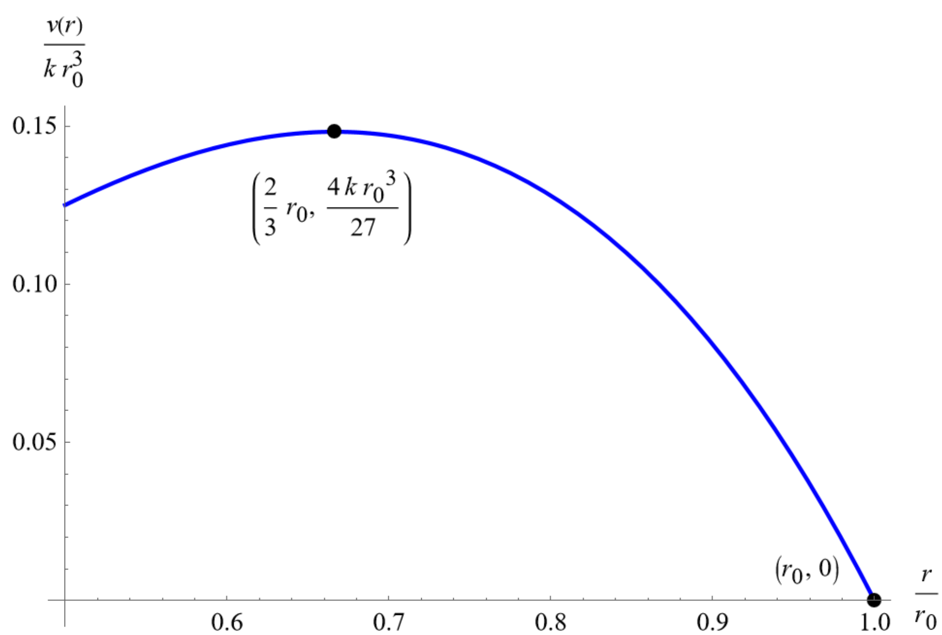
$$v\left(\frac{r_0}{2}\right) = k\left[r_0 - \left(\frac{1}{2}r_0\right)\right]\left(\frac{1}{2}r_0\right)^2 = \frac{kr_0^3}{8} = 0.125kr_0^3$$

$$v(r_0) = k(r_0 - r_0)r_0^2 = 0 \qquad \text{(absolute minimum)}$$

The smallest and largest of these numbers are the absolute minimum and maximum, respectively, over the interval $r_0/2 \le r \le r_0$. The maximum value of v occurs when $r = (2/3)r_0$, which is in agreement with the X-ray evidence. The maximum velocity is $4kr_0^3/27$. Rewrite the function in terms of dimensionless variables,

$$\begin{aligned} v(r) &= k(r_0 - r)r^2 \\ &= kr_0\left(1 - \frac{r}{r_0}\right)r^2 \\ &= kr_0^3\left(1 - \frac{r}{r_0}\right)\frac{r^2}{r_0^2} \\ \frac{v(r)}{kr_0^3} &= \left(1 - \frac{r}{r_0}\right)\left(\frac{r}{r_0}\right)^2, \end{aligned}$$

in order to graph it. Note that since $r_0/2 \le r \le r_0$, $1/2 \le r/r_0 \le 1$.



Below is a graph of the same function over $[0, r_0]$.